Radiation pressure on a dielectric boundary: was Poynting wrong?

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Abstract

When a plane electromagnetic wave in air falls on a flat dielectric boundary, the dielectric body is pulled toward the air as predicted by Poynting a century ago. According to Noether's theorem, the momentum in the direction parallel to the boundary must be conserved among the incident, reflected and transmitted waves. This uniquely determines the expression for the wave momentum in the dielectric medium which agrees with the Minkowski form. The inward force recently predicted and accompanied by Abraham's momentum are not consistent with Fresnel's formulae and basic symmetry principles.

The problem of momentum conservation in reflection and transmission of an electromagnetic wave entering (or leaving) a dielectric medium has been a matter of controversy over the past century. In 1905, Poynting [1] predicted that the dielectric is pulled toward air region. In this case, the momentum of the wave transmitted into the dielectric coincides with that proposed by Minkowski [2] which is also consistent with the classical relationship between the energy flux and momentum flux,

$$\text{energy flux} = \text{wave velocity} \ \times \ \text{momentum flux} \tag{1}$$

(We consider nondispersive waves since dispersion does not play important roles in the argument to follow.) Poynting even performed experiments using internal reflections of visible light in a glass slab and measured a torque exerted by the light wave in favorable agreement with the prediction. The outward force is also consistent with what is expected from the Maxwell's stress tensor. In the case of normal incidence, the electric field is tangential to the surface and thus should be continuous across the boundary. Then there is a pressure differential

$$\frac{1}{2}\left(\varepsilon-\varepsilon_0\right)E_t^2$$

acting from the dielectric region to air region (from higher energy density region to the lower). Momentum conservation among the three waves (incident, reflected , and transmitted) does not hold and the extra mechanical momentum, which is absorbed by the dielectric medium, exactly coincides with the pressure difference above [4]. Momentum non-conservation in wave reflection and transmission is not surprising and has been noticed in mechanical waves as well [7].

The validity of Poynting's analysis has recently been questioned by Loudon [5] who re-analyzed the momentum transfer using the Lorentz force arising from the polarization current. Without assuming a priori knowledge of the wave momentum in a dielectric, he concluded that the momentum absorbed by the dielectric medium is equal in magnitude but exactly opposite to the force predicted by Poynting. In Loudon's theory, the momentum carried by the wave in the dielectric turned out to be that proposed by Abraham [3] and smaller than that of Minkowski by a factor $n^2=\varepsilon/\varepsilon_0$ (in the case of nonmagnetic dielectric). Since the energy flux density is the same in both formulations because of energy conservation, the consequence is that the classical relationship in Eq. (1) has to be discarded in the Abraham's theory. Despite these difficulties in the Abraham's theory, the issue still remains

unsettled because no basic laws have been identified to uniquely define the momentum of electromagnetic waves in a dielectric medium. In some theory [8], an algebraic average of the two momenta has been claimed to be the wave momentum in dielectric materials. Moller [10] stated that "one is free to make a convenient choice since in any case the total moemtum density of fields and matter satsifies a conservation law." We believe there must be a basic law that uniquely determines the expression of the wave momentum in a dielectric which will prove Poynting was either correct or wrong.

In this Letter, we revisit the problem from an entirely different perspective exploiting the Noether theorem [6] which requires that momentum be conserved in the spatial direction of translational invariance. In the case of oblique incidence of electromagnetic wave on a dielectric medium, the Noether theorem tells us that the Poynting energy flux densities in the normal direction is conserved (consequence of invariance in the direction of time axis) and the momenta of the incident, reflected and transmitted waves parallel to the boundary surface are conserved. As will be shown, the latter constraint *uniquely* determines the momentum of electromagnetic waves in a dielectric medium which agrees with that of Minkowski. The parallel momentum conservation has not been considered or utilized (to our knowledge) in the past in terms of the fundamental symmetry properties of physical systems (Noether's theorem) which may have contributed to the ambiguity in defining wave momentum in a dielectric medium. The Fresnel's formulae [9] for reflected and transmitted waves are fully consistent with the Noether theorem and unique definition of the Minkowski momentum is actually hidden in the set of Fresnel's formulae.

Let us first review briefly the underlying problem by considering a very simple case: a plane electromagnetic wave having an amplitude E_0 in air falling normal on a flat surface of a semi-infinite, non-magnetic dielectric medium with an index of refraction n. (This is actually the geometry used in most studies.) The amplitudes of reflected and transmitted waves can be readily found from the continuity of electric and magnetic fields at the boundary and are given by

$$E_r = rE_0 = \frac{1-n}{1+n}E_0,\ \ E_t = (1+r)\,E_0 = \frac{2}{1+n}E_0 \tag{2}$$

The energy flux densities (Poynting vectors) of the incident, reflected and transmitted waves are well known,

$$S_i = c\varepsilon_0 E_0^2,\ \ S_r = \left(\frac{1-n}{1+n}\right)^2 c\varepsilon_0 E_0^2,\ \ S_t = \left(\frac{2n}{1+n}\right)^2 \frac{c}{n}\varepsilon_0 E_0^2 \tag{3}$$

It is evident that energy conservation holds $S_i = S_r + S_t$. However, momentum flux density is not conserved among the three waves. According to the classical relationship in Eq. (1), the momentum flux densities of the three waves are

$$P_i = \varepsilon_0 E_0^2,\ P_r = -\left(\frac{1-n}{1+n}\right)^2 \varepsilon_0 E_0^2,\ P_t = \left(\frac{2}{1+n}\right)^2 \varepsilon E_0^2 \tag{4}$$

The momentum conservation clearly does not hold among the three waves and the difference

$$\Delta P = P_i - P_r - P_t = \frac{2\left(n^2+1\right)}{\left(1+n\right)^2}\varepsilon_0 E_0^2 - \left(\frac{2n}{1+n}\right)^2 \varepsilon_0 E_0^2 = -\frac{2\left(n^2-1\right)}{(n+1)^2}\varepsilon_0 E_0^2 \tag{5}$$

must be acting on the dielectric body as a mechanical force. Since the electric field at the boundary $E_t = 2E_0/(1+n)$ is continuous, the difference in the energy densities exerts a pressure which exactly coincides with the extra momentum flux density,

$$\frac{1}{2}\left(\varepsilon_0 - \varepsilon\right) E_t^2 = \Delta P \tag{6}$$

It is noted that the momentum flux density adopted in the analysis here is that of Poynting-Minkowski. Loudon [5] showed an alternative partition,

$$P_i - P_r = \frac{2\left(n^2+1\right)}{\left(1+n\right)^2}\varepsilon_0 E_0^2 = \frac{4}{\left(1+n\right)^2}\varepsilon_0 E_0^2 + \frac{2\left(n^2-1\right)}{(n+1)^2}\varepsilon_0 E_0^2 \tag{7}$$

Here the partition is between the Abraham's momentum

$$P_A = \frac{4}{\left(1+n\right)^2}\varepsilon_0 E_0^2$$

and the "surface force"

$$\frac{2\left(n^2-1\right)}{(n+1)^2}\varepsilon_0 E_0^2 = \frac{1}{2}\left(\varepsilon - \varepsilon_0\right) E_t^2$$

which is inward. However, as will be shown, Abraham's momentum cannot satisfy the Noether theorem.

The Noether theorem pertains to conservation of energy and linear momentum in the direction parallel to the boundary. We therefore consider oblique incidence with incidence angle α and refraction angle α'. The angles α and α' are related through the Snell's law, $\sin\alpha = n\sin\alpha'$. If the magnetic field is in the incidence plane, the electric fields are parallel to the surface. For an incident electric field E_0, reflected and transmitted fields are

$$E_r = rE_0,\ E_t = \left(1+r\right) E_0$$

where r is the reflection coefficient for the assumed polarization

$$r = \frac{\sin(\alpha' - \alpha)}{\sin(\alpha' + \alpha)} \tag{8}$$

The Poynting fluxes associated with each wave are

$$\text{incident: } S_i = c\varepsilon_0 E_0^2; \text{ reflected: } S_r = r^2 c\varepsilon_0 E_0^2$$

$$\text{transmitted: } S_t = (1+r)^2 \frac{c}{n} n^2 \varepsilon_0 E_0^2 = (1+r)^2 cn\varepsilon_0 E_0^2$$

The normal components of the Poynting fluxes are conserved

$$(S_i - S_r)\cos\alpha = S_t \cos\alpha' \tag{9}$$

since for the reflection coefficient in Eq. (8), the following identity holds

$$(1-r)\cos\alpha = n\,(1+r)\cos\alpha' \tag{10}$$

This can be interpreted as energy flux (not density) along the direction of respective waves since $\cos\alpha$ factor essentially indicates the beam width. (See Fig. 1.) Then according to Eq. (1), the classical momentum fluxes (not flux density) of each wave are

$$\text{incident: } P_i = \varepsilon_0 E_0^2 \cos\alpha; \text{ reflected: } P_r = r^2 \varepsilon_0 E_0^2 \cos\alpha \tag{11}$$

$$\text{transmitted: } P_t = (1+r)^2 n^2 \varepsilon_0 E_0^2 \cos\alpha' \tag{12}$$

The components parallel to the boundary surface are

$$\text{incident: } \varepsilon_0 E_0^2 \cos\alpha \sin\alpha; \text{ reflected: } r^2 \varepsilon_0 E_0^2 \cos\alpha \sin\alpha \tag{13}$$

$$\text{transmitted: } (1+r)^2 n^2 \varepsilon_0 E_0^2 \cos\alpha' \sin\alpha' \tag{14}$$

It is evident that momentum conservation holds,

$$\varepsilon_0 E_0^2 \left(1 - r^2\right) \cos\alpha \sin\alpha = (1+r)^2 n^2 \varepsilon_0 E_0^2 \cos\alpha' \sin\alpha' \tag{15}$$

if Snell's law $\sin\alpha' = \sin\alpha/n$ and energy conservation in Eq. ([**?**]) are recalled. Therefore, Fresnel's reflection/transmission formulae fully satisfy the Noether theorem. Parallel momentum conservation is a consequence of energy conservation as well. From the last term in Eq. (15), it is concluded that the momentum flux density of electromagnetic wave in a dielectric medium is *uniquely* given by

$$n^2 \varepsilon_0 E_t^2 = \varepsilon E_t^2 \tag{16}$$

Other forms of momentum such as Abraham's momentum do not satisfy the basic theorem. The momentum agrees with Minkowski's definition of photon momentum in a dielectric and is also implicitly consistent with the negative pressure predicted by Poynting.

When the electric field is in the incidence plane, the reflected and transmitted waves are

$$E_r = \frac{\tan(\alpha - \alpha')}{\tan(\alpha + \alpha')} E_0, \;\; E_t = \frac{2\cos\alpha \sin\alpha'}{\sin(\alpha + \alpha')\cos(\alpha - \alpha')} E_0$$

In this case too, energy conservation

$$E_0^2 \left(1 - r^2\right) \cos\alpha = n E_t^2 \cos\alpha'$$

holds and we find $n^2 \varepsilon_0 E_t^2$ to be the wave momentum flux in the dielectric.

The Minkowski momentum is therefore arises from the translational invariance and ensuing conservation of momentum parallel to the boundary surface. The Noether theorem asserts that translational invariance of a physical system in a spatial direction implies momentum conservation in this direction. The Noether theorem also tells us that invariance under time translations implies energy conservation. For a stationary planar air-dielectric interface, we therefore must have momentum conservation parallel to the interface and energy conservation. The analyses for separate polarizations presented above can be unified as follows.

We consider incidence of an electromagnetic wave with incidence angle α with respect to the normal axis as shown in Fig. 1. The region $z < 0$ is assumed to be air with $n = 1$ and the region $z > 0$ is a medium with permittivity ε and permeability μ. The reflected and transmitted waves in terms of the incident fields $E_{i\|}$ and $E_{i\perp}$ with polarizations parallel and perpendicular to the incidence plane are

$$\frac{E_{r\|}}{E_{i\|}} = \frac{\sqrt{\varepsilon_0 \mu}\cos\alpha' - \sqrt{\varepsilon\mu_0}\cos\alpha}{\sqrt{\varepsilon_0 \mu}\cos\alpha' + \sqrt{\varepsilon\mu_0}\cos\alpha} \tag{17}$$

$$\frac{E_{t\|}}{E_{i\|}} = \frac{2\sqrt{\varepsilon_0 \mu}\cos\alpha}{\sqrt{\varepsilon_0 \mu}\cos\alpha' + \sqrt{\varepsilon\mu_0}\cos\alpha} \tag{18}$$

$$\frac{E_{r\perp}}{E_{i\perp}} = \frac{\sqrt{\varepsilon_0 \mu}\cos\alpha - \sqrt{\varepsilon\mu_0}\cos\alpha'}{\sqrt{\varepsilon_0 \mu}\cos\alpha + \sqrt{\varepsilon\mu_0}\cos\alpha'} \tag{19}$$

$$\frac{E_{t\perp}}{E_{i\perp}} = \frac{2\sqrt{\varepsilon_0 \mu}\cos\alpha}{\sqrt{\varepsilon_0 \mu}\cos\alpha + \sqrt{\varepsilon\mu_0}\cos\alpha'} \tag{20}$$

Energy conservation and the well known expression for the wave energy density $u = \varepsilon \mathbf{E}^2$ in an isotropic medium then imply

$$\varepsilon_0 \mathbf{E}_i^2 V = \varepsilon_0 \mathbf{E}_r^2 V + \varepsilon \mathbf{E}_t^2 V' \tag{21}$$

where

$$\mathbf{E}_i = \mathbf{E}_{i\|} + \mathbf{E}_{i\perp}, \mathbf{E}_r = \mathbf{E}_{r\|} + \mathbf{E}_{r\perp}, \mathbf{E}_t = \mathbf{E}_{t\|} + \mathbf{E}_{t\perp}$$

and

$$\frac{V'}{V} = \frac{\cos\alpha'}{n\cos\alpha} = \frac{1}{n}\sqrt{\frac{1-\frac{1}{n^2}\sin^2\alpha}{1-\sin^2\alpha}} \tag{22}$$

is the volume conversion factor which results from the dilation of the beam width $\cos\alpha'/\cos\alpha$ upon entry to the dielectric and longitudinal compression of the incident beam due to the velocity ratio $c'/c = 1/n$. The condition for energy conservation between the incident, reflected and transmitted beams therefore reads

$$\varepsilon_0\left(\mathbf{E}_i^2 - \mathbf{E}_r^2\right)\cos\alpha = \frac{1}{n}\varepsilon\mathbf{E}_t^2\cos\alpha' \tag{23}$$

which is equivalent to the conservation of normal components of the Poynting vectors as noted earlier. It is easily confirmed from Eqs. (17) - (20) that the condition is identically fulfilled for arbitrary incidence angle α and polarization, as was of course expected. However, we have to recall that this little calculation is to confirm the volume conversion factor in Eq. (22), and to remind the reader that $u_t = \varepsilon\mathbf{E}_t^2$ is considered as the energy density in the transmitted wave in the dielectric without (to our knowledge) anybody ever questioning whether this would have to be split into a genuine electromagnetic component and a co-moving material component, or whether it would have to be denoted as a "pseudo-energy density" of the transmitted wave.

Another well known statement of energy conservation between the incident, reflected and transmitted beams results from Eq. (23) if we take into account the relation between energy density and energy flux density (Poynting flux) in a medium with refractive index n,

$$|\mathbf{S}| = \frac{c}{n}\varepsilon\mathbf{E}^2$$

Multiplying Eq. (21) by c yields

$$(S_i - S_r)\cos\alpha = S_t\cos\alpha'$$

Multiplying further by $\sin\alpha$, we obtain

$$(S_{ix} - S_{rx})\cos\alpha = nS_{tx}\cos\alpha'$$

where S_{jx} is the x component of each Poynting flux. Multiply by $V/\left(c^2\cos\alpha\right) = nV'/\left(c^2\cos\alpha'\right)$ to obtain

$$\frac{1}{c^2}\left(S_{ix} - S_{rx}\right)V = \frac{n^2}{c^2}S_{tx}V' \tag{24}$$

This asserts that the momentum density of the transmitted wave in the dielectric medium is uniquely determined as

$$\mathbf{p} = \frac{n^2}{c^2}\mathbf{S} \tag{25}$$

which is the Minkowski's formula.

In conclusion, in wave reflection and transmission of electromagnetic wave at a dielectric boundary, the wave momentum along the boundary surface must be conserved according to Noether's theorem. The Fresnel's formula are consistent with this basic requirement. The momentum flux density and momentum of electromagnetic wave in a dielectric medium uniquely takes the Minkowski's form without room for alternatives. The classical relationship between the energy flux density and momentum flux density in Eq. (1) is preserved to our comfort. The force to act on a dielectric when a wave enters it is outward as predicted and observed by Poynting, and as also required from Maxwell's stress tensor.

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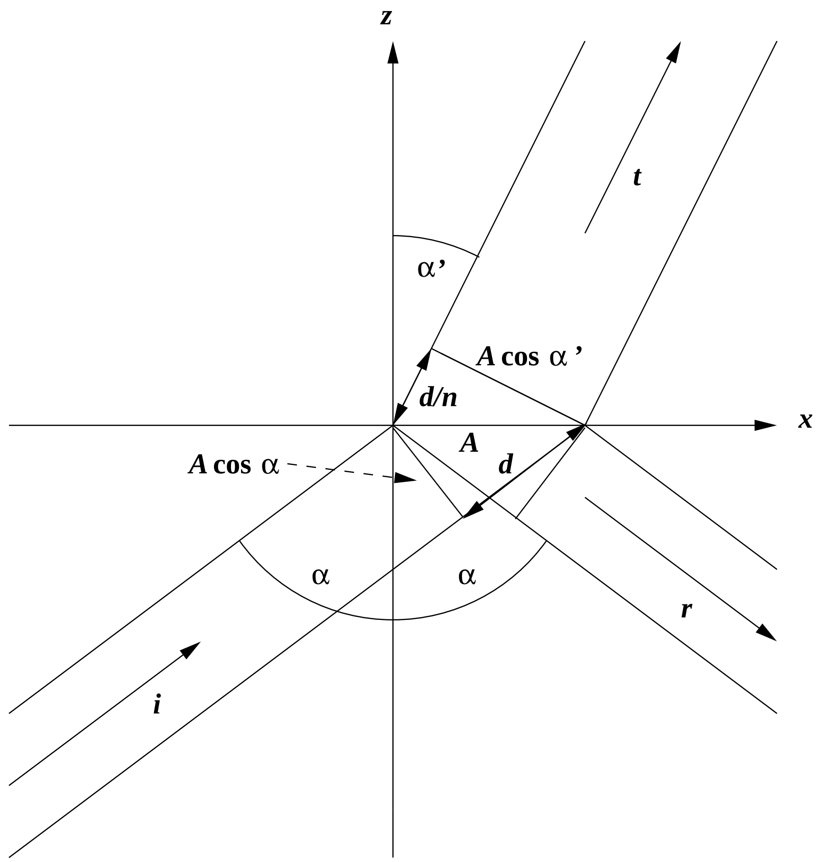

FIG. 1: The lower half space $z < 0$ is vacuum or air with dielectric parameters ϵ_0 and μ_0. The upper half space $z > 0$ is dielectric material with parameters ϵ and μ. The letters i, r, and t denote the incoming, reflected and transmitted beam. The beam cross sections $A \cos \alpha$ and $A \cos \alpha'$, and the lengths d and $d' = d/n$ are displayed to explain the volume conversion factor V'/V between the incoming and transmitted waves.